MATH 3060 Tutorial 3

Chan Ki Fung

September 28, 2022

1. Let f be a 2π periodic function, integrable on $[-\pi, \pi]$. Suppose further f is differentiable at $x = 0$, analyse the proof of theorem 1.5 in lecture note and show that
$$S_n(f)(0) \to f(0)$$
as $n \to \infty$. (Note that you cannot apply theorem 1.5 directly, because the condition here does not imply f is Lipschitz continuous at $x = 0$.)

2. Last time we show that
$$\frac{x^2}{2} - \frac{x}{2} + \frac{1}{12} = \frac{1}{2\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \cos(2\pi n x).$$
for $x \in [0, 1]$. In particular, we can put $x = 0$ and get
$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$
Now, re-derive this identity using the fact we learnt in tutorial 1:
$$x - \frac{1}{2} \sim -\frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin(2n\pi x)$$

3. Show that there exists no Riemann integrable functions f on $[-\pi, \pi]$ so that
   (a) $a_n(f) = 1$ for any $n \in \mathbb{Z}$.
   (b) $b_n(f) = \frac{1}{\sqrt{n}}$.
   (c) (Optional) $c_n(f) = \frac{1}{n}$ for $n > 0$ and $c_n(f) = 0$ for $n \leq 0$.

4. Let $r(t) = (f(t), g(t)), x \in [-\pi, \pi]$ be a simple closed curve in $\mathbb{R}^2$. Suppose $|r'(t)| \equiv 1$.
   (a) Show that the area A of the region bounded by r is
   $$\pi \left| \sum_{n=-\infty}^{\infty} \left(c_n(f)\overline{c_n(g')} - c_n(g)\overline{c_n(f')} \right) \right|$$
   (Hint: $A = \frac{1}{2} \left| \int_r x dy - y dx \right|$ by Green's theorem.)

(b) Show that $A \leq \pi$.

5. (a) Fix $p \in (0,1)$, let $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be the function

$$d((x,y),(x',y')) = ((x-x')^p + (y-y')^p)^{\frac{1}{p}}.$$

Show that d is not a metric.

(b) the link Q2 for solution.) Let p be a prime number, consider the following function $N_p : \mathbb{Q} \to \mathbb{R}$. Each nonzero rational number x can be written in the form

$$x = p^n \frac{a}{b}$$

with n an integer, and a, b are integers not divisible by p. We define $N_p(x) = p^{-n}$, and also define $N_p(0) = 0$.
Show that $d(x,y) = N_p(x-y)$ is a metric on $\mathbb{Q}$.

(Optional, but see